

On a Griffiths-Harris Conjecture

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1 Introduction

In 1882, M. Noether claimed the following statement which was later proven by Lefschetz: For $d \geq 4$, a very general smooth degree d surface X in \mathbb{P}^3 has Picard number $\rho(X) = 1$. This motivates the definition of the *Noether-Lefschetz locus*, denoted by NL_d parametrizing the space of smooth degree d surfaces X in \mathbb{P}^3 with $\rho(X) > 1$. One of the interesting problems is to understand the geometry of the Noether-Lefschetz locus. By the Lefschetz $(1, 1)$ -theorem, we can look at an irreducible component of the Noether-Lefschetz locus locally as a Hodge locus (see [Voi03, §5] for more details). In particular, denote by $U_d \subseteq \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)))$ the open subscheme parametrizing smooth projective hypersurfaces in \mathbb{P}^3 of degree d . Let $\mathcal{X} \xrightarrow{\pi} U_d$ be the corresponding universal family. For a given $F \in U_d$, denote by X_F the surface $X_F := \pi^{-1}(F)$. Let $X \in U_d$ and $U \subseteq U_d$ be a simply connected neighbourhood of X in U_d (under the analytic topology). Then, $\pi|_{\pi^{-1}(U)}$ induces a variation of Hodge structure (\mathcal{H}, ∇) on U , where $\mathcal{H} := R^2\pi_*\mathbb{Z} \otimes \mathcal{O}_U$ and ∇ is the Gauss-Manin connection. Note that \mathcal{H} defines a local system on U whose fiber over a point $F \in U$ is $H^2(X_F, \mathbb{Z})$. Consider a non-zero element $\gamma_0 \in H^2(X_F, \mathbb{Z}) \cap H^{1,1}(X_F, \mathbb{C})$ such that $\gamma_0 \neq c_1(\mathcal{O}_{X_F}(k))$ for $k \in \mathbb{Z}_{>0}$. This defines a section $\gamma \in (\mathcal{H} \otimes \mathbb{C})(U)$. Let $\bar{\gamma}$ be the image of γ in $\mathcal{H}/F^2(\mathcal{H} \otimes \mathbb{C})$. The Hodge loci corresponding to γ ,

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denoted $\text{NL}(\gamma)$ is then defined as

$$\text{NL}(\gamma) := \{G \in U \mid \overline{\gamma}_G = 0\},$$

where $\overline{\gamma}_G$ denotes the value at G of the section $\overline{\gamma}$. For an irreducible component $L \subset \text{NL}_d$ and $X \in L$, general, we can find $\gamma \in H^{1,1}(X, \mathbb{Z}) := H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{C})$ such that $\overline{\text{NL}(\gamma)} = L$ (the closure taken in U_d under Zariski topology).

One of the first results in this direction is due to Green, Griffiths, Voisin and others ([Gre89, GH83, Voi88]) which states that for an irreducible component L of the Noether-Lefschetz locus, that for $d \geq 4$,

$$d - 3 \leq \text{codim}(L, U_d) \leq \binom{d-1}{3}.$$

The upper bound follows easily from the fact that $\dim H^{2,0}(X) = \binom{d-1}{3}$ for any $X \in U_d$ (see [Voi03, §6]). We say that L is a *general* component if $\text{codim } L = \binom{d-1}{3}$ and *special* otherwise. It was proven by Ciliberto, Harris and Miranda [CHM88] that for $d \geq 4$, the Noether-Lefschetz locus has infinitely many general components and the union of these components is Zariski dense in U_d . The guiding principle of much work in the area has been the expectation that special components should be due to the presence of low degree curves. Voisin [Voi89] and Green [Gre89] independently proved that for $d \geq 5$, $\text{codim } L = d - 3$ if and only if L parametrizes surfaces of degree d containing a line. If $d - 3 < \text{codim } L \leq 2d - 7$ then $\text{codim } L = 2d - 7$ and L parametrizes the surfaces containing a conic. Otwinowska [Otw03] proved that for an integer $b > 0$ and $d \gg b$ if $\text{codim } L \leq bd$ then L parametrizes surfaces containing a curve of degree at most b .

For $r \geq 3$, we define the level r -Noether-Lefschetz locus, denoted $\text{NL}_{r,d}$ to be the space parametrizing surfaces with Picard number greater than or equal to r . It has been conjectured by Griffiths and Harris in [GH83] that for $r < d$, an irreducible component of $\text{NL}_{r,d}$ is of codimension greater than or equal to $(r-1)(d-3) - \binom{r-3}{2}$. Furthermore, the component of $\text{NL}_{r,d}$ parametrizing surfaces containing $r-1$ coplanar lines is of this codimension. In this article we prove (in Theorem 5.11) that:

Theorem 1.1. Let $r \geq 3$ and $d \gg r$. Let L be an irreducible component of $\text{NL}_{r,d}$. Then, $\text{codim } L \geq (r-1)(d-3) - \binom{r-3}{2}$. Furthermore, there exists a component L of $\text{NL}_{r,d}$ of this

codimension parametrizing surfaces containing $r - 1$ coplanar lines.

The techniques used to prove this result is a combination of deformation theory and Hodge theory. Instead of looking at the Hodge locus corresponding to a Hodge class, we study the Hodge locus corresponding to a \mathbb{Z} -module of Hodge classes. We then use a result due to Otwinowska, [Otw04, Theorem 1], to show that if the codimension of an irreducible component L of $\text{NL}_{r,d}$ is less than or equal to $(r - 1)(d - 3) - \binom{r-3}{2}$, then for a general $X \in L$ there exists a lattice $\Lambda \subset H^{1,1}(X, \mathbb{Z})$ generated by classes of curves of degree less than or equal to $r - 1$ such that L is locally of the form $\text{NL}(\Lambda)$ (see Proposition 5.6), where $\text{NL}(\Lambda)$ is the intersection of $\text{NL}(\gamma)$ for all $\gamma \in \Lambda$, $\bigcap_{\gamma \in \Lambda} \text{NL}(\gamma)$.

We now use the theory of semi-regularity as introduced in [Blo72] to reduce the problem to a question in flag Hilbert schemes. First to fix some notations, for a Hilbert polynomial P for some curve C in \mathbb{P}^3 , we denote by H_P the corresponding Hilbert scheme, parametrizing curves (schemes with pure dimension 1) with Hilbert polynomial P . Throughout this article we denote by Q_d the Hilbert polynomial of a degree d surface in \mathbb{P}^3 . We denote by H_{P,Q_d} the corresponding flag Hilbert scheme parametrizing pairs (C, X) such that $C \in H_P, X \in H_{Q_d}$ and $C \subset X$. A curve C on a smooth surface in \mathbb{P}^3 is said to be *semi-regular* if $H^1(\mathcal{O}_X(C)) = 0$. We prove that

Theorem 1.2. Let X be a smooth surface in \mathbb{P}^3 of degree d and $C \subset X$, a semi-regular curve with Hilbert polynomial, say P . For any irreducible component, L' of $\overline{\text{NL}([C])}$ (the closure is taken in U_d under Zariski topology) there exists an irreducible component H' of H_{P,Q_d} containing the pair (C, X) such that $\text{pr}_2(H')_{\text{red}}$ coincides with L'_{red} , where pr_2 is the second projection map from H_{P,Q_d} to H_{Q_d} . In particular, if C is reduced, connected and $d \geq \deg(C) + 4$ then this holds true and the irreducible component H' is uniquely determined by L' .

See Theorem 3.10 and Lemma 3.4 for the precise statements and its proof. Using this proof we further show that under the above bound on the codimension of L , the lattice will infact be generated by classes of lines (see Lemma 5.9). Finally, we do a computation in Proposition 5.10, to determine the “arrangement” of these lines which would help us determine the component with the correct codimension.

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2 Introduction to Noether-Lefschetz locus

2.1. In this section we recall the basic definitions of Noether-Lefschetz locus. See [Voi02, §9, 10] and [Voi03, §5, 6] for a detailed presentation of the subject.

Notation 2.2. By a *component* of NL_d , we mean an irreducible component. By a *surface* we always mean a smooth surface in \mathbb{P}^3 . Denote by Q_d the Hilbert polynomial of degree d surfaces in \mathbb{P}^3 . Given, a Hilbert polynomial P , denote by H_P the corresponding Hilbert scheme and by H_{P, Q_d} the corresponding flag Hilbert scheme. Also, for a point $u \in U_d$, denote by X_u the fiber $\pi^{-1}(u)$.

Notation 2.3. Let $X \in U_d$ and $\mathcal{O}_X(1)$, the very ample line bundle on X determined by the closed immersion $X \hookrightarrow \mathbb{P}^3$ arising (as in [Har77, II.Ex.2.14(b)]) from the graded homomorphism of graded rings $S \rightarrow S/(F_X)$, where $S = \Gamma_*(\mathcal{O}_{\mathbb{P}^3})$ and F_X is the defining equations of X . Denote by H_X the very ample line bundle $\mathcal{O}_X(1)$. Note that a very ample line bundle on X_u for any $u \in U$ remains very ample in the family \mathcal{X} , hence the corresponding cohomology class remains of type $(1, 1)$ in \mathcal{X} .

2.4. Let X be a surface. The Lefschetz hyperplane section theorem implies,

$$H^2(X, \mathbb{C}) \cong H^2(X, \mathbb{C})_{\text{prim}} \oplus \mathbb{C}H_X,$$

where H_X is the very ample line bundle on X and $H^2(X, \mathbb{C})_{\text{prim}}$ is the primitive cohomology. This gives us a natural projection map from $H^2(X, \mathbb{C})$ to $H^2(X, \mathbb{C})_{\text{prim}}$. For $\gamma \in H^2(X, \mathbb{C})$, denote by γ_{prim} the image of γ under this morphism. Since the very ample line bundle H_X remains of type $(1, 1)$ in the family \mathcal{X} , we can therefore conclude that $\gamma \in H^{1,1}(X)$ remains of type $(1, 1)$ if and only if γ_{prim} remains of type $(1, 1)$. In particular, $\text{NL}(\gamma) = \text{NL}(\gamma_{\text{prim}})$.

Definition 2.5. We now discuss the tangent space to the Hodge locus, $\text{NL}(\gamma)$. We know that the tangent space to U at X , $T_X U$ is isomorphic to $H^0(\mathcal{N}_{X|\mathbb{P}^3})$. This is because U is an open subscheme of the Hilbert scheme H_{Q_d} , the tangent space of which at the point X is simply $H^0(\mathcal{N}_{X|\mathbb{P}^3})$. Given the variation of Hodge structure above, we have (by Griffith's transversality) the differential map:

$$\overline{\nabla} : H^{1,1}(X) \rightarrow \text{Hom}(T_X U, H^2(X, \mathcal{O}_X))$$

induced by the Gauss-Manin connection. Given $\gamma \in H^{1,1}(X)$ this induces a morphism, denoted $\overline{\nabla}(\gamma)$ from $T_X U$ to $H^2(\mathcal{O}_X)$. The tangent space at X to $\text{NL}(\gamma)$ is then defined to be $\ker(\overline{\nabla}(\gamma))$.

2.6. The boundary map

$$\rho : H^0(\mathcal{N}_{X|\mathbb{P}^3}) \rightarrow H^1(\mathcal{T}_X)$$

arising from the long exact sequence associated to the short exact sequence:

$$0 \rightarrow \mathcal{T}_X \rightarrow \mathcal{T}_{\mathbb{P}^3}|_X \rightarrow \mathcal{N}_{X|\mathbb{P}^3} \rightarrow 0$$

is called the *Kodaira-Spencer* map. The morphism $\overline{\nabla}(\gamma)$ is related to the Kodaira-Spencer map as we will see below.

2.7. Note that there exists a natural cup product morphism,

$$H^1(X, \mathcal{T}_X) \otimes H^1(X, \Omega_X^1) \xrightarrow{\cup} H^2(X, \mathcal{O}_X).$$

For $\gamma \in H^1(\Omega_X^1)$ this induces a morphism, denoted $\cup \gamma$, from $H^1(\mathcal{T}_X)$ to $H^2(\mathcal{O}_X)$. We then have the following result in Hodge theory (see [Voi02, Theorem 10.21]):

Lemma 2.8. The differential map $\overline{\nabla}(\gamma)$ coincides with the following:

$$T_X U \cong H^0(\mathcal{N}_{X|\mathbb{P}^3}) \xrightarrow{\rho} H^1(\mathcal{T}_X) \xrightarrow{\cup \gamma} H^2(\mathcal{O}_X).$$

3 Hodge locus and Hilbert flag schemes

3.1. In this section we define what is a semi-regular map. We then briefly study Hodge locus for a family of smooth projective surfaces in \mathbb{P}^3 and show how it is related to certain Hilbert flag schemes. More specifically, we shall study the Hodge locus corresponding to certain effective algebraic cycles which will be semi-regular. For such classes we will see that the Hodge locus “coincides” with a component of a flag Hilbert scheme. We elaborate on the details in this section.

3.1 Semi-regularity map and tangent space to Hodge locus

3.2. We start with the definition of a semi-regular curve. Let X be a surface and $C \subset X$, a curve in X . Since X is smooth, C is local complete intersection in X . This gives rise to the short exact sequence:

$$0 \rightarrow \mathcal{O}_X(-C) \rightarrow \mathcal{O}_X \rightarrow i_* \mathcal{O}_C \rightarrow 0,$$

where i is the natural inclusion morphism from C into X . Note that, $\mathcal{O}_X(C)$ is locally free \mathcal{O}_X -module, hence flat. Therefore, tensoring this short exact sequence by $\mathcal{O}_X(C)$ we get

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(C) \rightarrow \mathcal{N}_{C|X} \rightarrow 0 \tag{1}$$

is exact, where $\mathcal{N}_{C|X}$ is the normal sheaf $\mathcal{H}om_X(\mathcal{O}_X(-C), i_* \mathcal{O}_C)$ which is isomorphic to the sheaf $i_* \mathcal{O}_C \otimes_{\mathcal{O}_X} \mathcal{O}_X(C)$ (see [Har77, Ex. II.5.1(b)]). The *semi-regularity map* is the morphism

$$\pi : H^1(\mathcal{N}_{C|X}) \rightarrow H^2(\mathcal{O}_X)$$

which arises from the long exact sequence associated to the short exact sequence (1). We say that C is *semi-regular* if π is injective.

3.3. The Lefschetz hyperplane section theorem implies that $H^1(\mathcal{O}_X) = 0$. Then, the long exact sequence associated to (1) contains the following segment:

$$0 \rightarrow H^1(\mathcal{O}_X(C)) \rightarrow H^1(\mathcal{N}_{C|X}) \xrightarrow{\pi} H^2(\mathcal{O}_X).$$

So, $H^1(\mathcal{O}_X(C)) = 0$ is equivalent to π being injective, hence C being semi-regular. We now prove a result that would help us determine when a curve is semi-regular.

Lemma 3.4. Let C be a connected reduced curve and $d \geq \deg(C) + 4$ then $h^1(\mathcal{O}_X(C)) = 0$. In particular, C is semi-regular.

Proof. Since X is a hypersurface in \mathbb{P}^3 of degree d , $\mathcal{I}_X \cong \mathcal{O}_{\mathbb{P}^3}(-d)$. Consider the short exact sequence:

$$0 \rightarrow \mathcal{I}_X \rightarrow \mathcal{I}_C \rightarrow \mathcal{O}_X(-C) \rightarrow 0.$$

Tensoring this by $\mathcal{O}_{\mathbb{P}^3}(k)$, we get the following terms in the associated long exact sequence:

$$\dots \rightarrow H^1(\mathcal{I}_C(k)) \rightarrow H^1(\mathcal{O}_X(-C)(k)) \rightarrow H^2(\mathcal{I}_X(k)) \rightarrow \dots$$

Now, $H^2(\mathcal{O}_{\mathbb{P}^3}(k-d)) = 0$ (see [Har77, Theorem 5.1]) and \mathcal{I}_C is $\deg(C)$ -regular (see [Gia06, Main Theorem]). So, $H^1(\mathcal{I}_C(k)) = 0$ for $k \geq \deg(C)$. This implies $H^1(\mathcal{O}_X(-C)(k)) = 0$ for $k \geq \deg(C)$. By Serre duality, $0 = H^1(\mathcal{O}_X(-C)(d-4)) \cong H^1(\mathcal{O}_X(C))$. So, C is semi-regular. \square

3.5. Let X be a surface and $C \subset X$ be a curve. We now do a computation to show that for $d \geq \deg(C) + 4$, $\dim |C| = 0$, where $|C|$ is the linear system of C in X .

Lemma 3.6. Let $d \geq 5$ and C be an effective divisor on a smooth degree d surface X of the form $\sum_i a_i C_i$, where C_i are integral curves with $\deg(C) + 4 \leq d$. Then, $h^0(\mathcal{N}_{C|X}) = 0$. In particular, $\dim |C| = 0$, where $|C|$ is the linear system associated to C .

Proof. Let $C = \sum_i a_i C_i$ with C_i integral. Then,

$$\deg((\mathcal{O}_X(C)|_C \otimes \mathcal{O}_C)|_{C_i}) = a_i C_i^2 + \sum_{j \neq i} a_j C_i \cdot C_j.$$

Denote by $e_i := \deg(C_i)$. Using the adjunction formula and the fact that $K_X \cong \mathcal{O}_X(d-4)$, we have that

$$\begin{aligned} \deg((\mathcal{O}_X(C)|_C \otimes \mathcal{O}_C)|_{C_i}) &= 2a_i \rho_a(C_i) - 2a_i - (d-4)a_i e_i + \sum_{j \neq i} a_j C_i \cdot C_j \\ &\leq a_i(e_i^2 - (d-1)e_i) + \sum_{j \neq i} a_j C_i C_j \\ &\leq a_i(e_i^2 - 3e_i - e_i \sum_j a_j e_j) + \sum_{j \neq i} a_j e_i e_j. \end{aligned}$$

The first inequality follows from the bound on the genus of a curve in \mathbb{P}^3 in terms of its degree (see [Har77, Example 6.4.2]). The second inequality follows from the facts that $d \geq \deg(C) + 4$ and $C_i \cdot C_j \leq e_i e_j$. It then follows directly that $\deg((\mathcal{O}_X(C)|_C \otimes \mathcal{O}_C)|_{C_i}) < 0$. This implies that $h^0(C_i, (\mathcal{O}_X(C)|_C \otimes \mathcal{O}_C)|_{C_i}) = 0$ for all i . So, $h^0(\mathcal{N}_{C|X}) = h^0(C, \mathcal{O}_X(C)|_C \otimes \mathcal{O}_C) = 0$.

Since $h^1(\mathcal{O}_X) = 0$ (by Lefschetz hyperplane section Theorem) and $h^0(\mathcal{O}_X) = 1$, using the

long exact sequence associated to the short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(C) \rightarrow \mathcal{O}_X(C)|_C \otimes \mathcal{O}_C \rightarrow 0 \quad (2)$$

we get that $h^0(\mathcal{O}_X(C)) = 1$. Since $|C| = \mathbb{P}(H^0(\mathcal{O}_X(C)))$, the lemma follows. \square

3.2 Flag Hilbert scheme and Hodge locus

3.7. In this section we introduce the basic definitions of flag Hilbert schemes. See [Ser06, §4] for further details. We then prove the main result of this section which relates Hodge locus to Hilbert schemes.

3.8. Given an m -tuple of polynomials $\mathcal{P}(t) = (P_1(t), P_2(t), \dots, P_m(t))$, we define the contravariant functor, called the *Hilbert flag functor* relative to $\mathcal{P}(t)$,

$$FH_{\mathcal{P}(t)} : (\text{schemes}) \rightarrow \text{sets}$$

$$S \mapsto \{(X_1, X_2, \dots, X_m) \mid X_1 \subset X_2 \subset \dots \subset \mathbb{P}_S^3\}$$

such that the Hilbert polynomial of X_i is $P_i(t)$ and X_i is an S -closed subscheme of X_{i+1} . We call such an m -tuple a *flag relative to $\mathcal{P}(t)$* .

3.9. The functor $FH_{\mathcal{P}(t)}$ is representable by a projective scheme, $H_{\mathcal{P}(t)}$ which parametrizes all such flags relative to $\mathcal{P}(t)$. We call this the *Hilbert flag scheme*.

Theorem 3.10. Let X be a surface, C be a semi-regular curve in X and $\gamma \in H^{1,1}(X, \mathbb{Z})$ be the class of the curve C . For any irreducible component L' of $\overline{\text{NL}(\gamma)}$ (the closure is taken in the Zariski topology on U_d) there exists an irreducible component H' of H_{P, Q_d} containing the pair (C, X) such that the associated reduced scheme $\text{pr}_2(H')_{\text{red}}$ coincides with L'_{red} , where pr_2 is the second projection map from H_{P, Q_d} to H_{Q_d} . Furthermore, if $d \geq \deg(C) + 4$ then such H' is uniquely determined by L' .

Proof. The first part of the theorem follows directly from [Blo72, Theorem 7.1].

Furthermore, Lemma 3.6 implies that $\dim |C| = 0$. So, given an irreducible component, say L' of $\overline{\text{NL}(\gamma)}$ such that X is a general element, there exists a unique irreducible component H'

of H_{P,Q_d} containing the pair (C, X) such that $\text{pr}_2(H')_{\text{red}}$ coincides with L'_{red} . This proves the rest of the theorem. □

4 Variation of lattices

4.1. In this section we give a formula to compute the dimension of an irreducible component of a Hodge locus (see Proposition 4.6). This result will be particularly useful to prove the asymptotic case of a Griffiths-Harris conjecture, which we see in the next section.

4.2. Let X be a surface of degree d . An *augmented lattice* Λ_X on X of rank r , is a rank r \mathbb{Z} -submodule $\Lambda_X \subset H^2(X, \mathbb{Z})$ generated by the class of the very ample line bundle H_X (as in 2.3) and cohomology classes of $r - 1$ reduced curves, say C_1, \dots, C_{r-1} such that Λ_X is saturated in the sense that for all $\lambda \in \Lambda_X, c \in \mathbb{Q}$ if $c\lambda \in H^2(X, \mathbb{Z})$ then $c\lambda \in \Lambda_X$. For such Λ_X , we say that C_i for $i = 1, \dots, r - 1$ *generate* Λ_X . We say that Λ_X is *prime* if C_1, \dots, C_{r-1} are integral.

4.3. Let Λ_X be as in 4.2. We can define

$$\text{NL}(\Lambda_X) := \{G \in U \mid \overline{\gamma}_G = 0, \text{ for all } \gamma \in \Lambda_X\}.$$

For a surface X and reduced curve $C \subset X$ such that the cohomology class $[C]$ of C is not a \mathbb{Q} -multiple of $c_1(H_X)$, denote by Λ_X^0 the rank 2 \mathbb{Z} -module generated by $[C]$ and $c_1(H_X)$, where c_1 is the first Chern class map. Since a very ample line bundle remains of type $(1, 1)$ in the family \mathcal{X} , $\text{NL}(\Lambda_X^0)$ coincides with $\text{NL}([C])$. More generally, for a rank r augmented lattice Λ_X generated by C_1, \dots, C_{r-1} we have, $\text{NL}(\Lambda_X)$ is isomorphic to the fiber product $\text{NL}([C_1]) \times_{H_{Q_d}} \dots \times_{H_{Q_d}} \text{NL}([C_{r-1}])$.

4.4. Let Λ_X be as before of rank 2, generated by a reduced curve, say C . Let P be the Hilbert polynomial of C . Assume $d \geq \deg(C) + 4$. Using Theorem 3.10 we can conclude that for general $X' \in \text{NL}(\Lambda_X)$ there exists a curve $C' \subset X'$ such that $\overline{\text{NL}([C'])}$ is an irreducible component of $\overline{\text{NL}(\Lambda_X)}$ and C' deforms to C , i.e., C' has the same Hilbert polynomial P . Denote by $\Lambda_{X'}$ the augmented lattice on X' of rank 2 generated by C' . Theorem 3.10 again implies that there exists

an unique irreducible component, denoted $H_{\Lambda_{X'}}$, of H_{P,Q_d} containing the pair (C', X') such that $\text{pr}_2(H_{\Lambda_{X'}})$ is isomorphic to $\overline{\text{NL}(\Lambda_{X'})}$. Denote by $L_{\Lambda_{X'}} := \text{pr}_1(H_{\Lambda_{X'}})$. From now on, we will always assume that $\overline{\text{NL}(\Lambda_X)}$ is irreducible, which is equivalent to X being general in $\text{NL}(\Lambda_X)$, in particular, away from the points of intersection of any two irreducible components of $\overline{\text{NL}(\Lambda_X)}$.

4.5. Suppose now that Λ_X is of rank r generated by C_1, \dots, C_{r-1} . Let P_1, \dots, P_{r-1} be the Hilbert polynomials of C_1, \dots, C_{r-1} , respectively. Consider the natural morphism

$$p : H_{P_1, Q_d} \times_{H_{Q_d}} \dots \times_{H_{Q_d}} H_{P_{r-1}, Q_d} \rightarrow H_{Q_d}.$$

Assume $d \geq \sum_{i=1}^{r-1} \deg(C_i) + 4$. Using Theorem 3.10, we can conclude that for every irreducible component L' of $\overline{\text{NL}(\Lambda_X)}$ there exists an unique irreducible component, say H of $H_{P_1, Q_d} \times_{H_{Q_d}} \dots \times_{H_{Q_d}} H_{P_{r-1}, Q_d}$ containing $(C_1, X) \times \dots \times (C_{r-1}, X)$ such that $p(H)$ coincides with L' . Similarly as in 4.4, by taking X general in $\text{NL}(\Lambda_X)$ we can ensure that $\overline{\text{NL}(\Lambda_X)}$ is irreducible. Denote by H_{Λ_X} the irreducible component of $H_{P_1, Q_d} \times_{H_{Q_d}} \dots \times_{H_{Q_d}} H_{P_{r-1}, Q_d}$ such that $p(H_{\Lambda_X})$ coincides with $\overline{\text{NL}(\Lambda_X)}$. Denote by $L_{\Lambda_X} := \text{pr}(H_{\Lambda_X})$, where pr is the natural projection map from $H_{P_1, Q_d} \times_{H_{Q_d}} \dots \times_{H_{Q_d}} H_{P_{r-1}, Q_d}$ to $H_{P_1} \times \dots \times H_{P_{r-1}}$.

Proposition 4.6. Let $r \geq 3$, X be a surface of degree d and Λ_X be an augmented lattice of rank $r + 1$ generated by r reduced curves C_1, \dots, C_r . Assume that $\sum_{i=1}^r \deg(C_i) + 4 \leq d$. Then, the dimension of $\overline{\text{NL}(\Lambda_X)}$ is given by the following formula:

$$\text{codim } \overline{\text{NL}(\Lambda_X)} = \text{codim } I_d(C) - \dim L_{\Lambda_X}$$

where $C = C'_1 \cup \dots \cup C'_r$ for a general r -tuple (C'_1, \dots, C'_r) in L_{Λ_X} and $I_d(C)$ is the degree d graded piece in the ideal $I(C)$ of C .

Proof. Consider the diagram,

$$\begin{array}{ccc} H_{\Lambda_X} & \xrightarrow{\text{pr}_1} & L_{\Lambda_X} \\ \text{pr}_2 \downarrow & & \\ \overline{\text{NL}(\Lambda_X)} & & \end{array}$$

Denote by P_i the Hilbert polynomial of curves C_i , respectively. Recall, L_{Λ_X} is contained in

$H_{P_1} \times \dots \times H_{P_r}$. For an r -tuple $(C_1, \dots, C_r) \in L_{\Lambda_X}$, the fiber of pr_1 parametrizes the space of smooth degree d surfaces containing $C = C_1 \cup \dots \cup C_r$, which is an open subscheme in $\mathbb{P}(I_d(C))$. Since $I_d(C)$ is irreducible, the dimension of the generic fiber of pr_1 is equal to $\dim I_d(C) - 1$, where $(C_1, \dots, C_r) \in L_{\Lambda_X}$ is a general element. The fiber of pr_2 over $\text{pr}_2((C_1, \dots, C_r, X))$ is isomorphic to $|C_1| \times \dots \times |C_r|$. But, Lemma 3.6 implies $\dim |C_i| = 0$ for $i = 1, \dots, r$. So, the dimension of the generic fiber of pr_2 is zero. Then,

$$\dim H_{\Lambda_X} = \dim L_{\Lambda_X} + \dim I_d(C) - 1 = \dim \text{NL}(\Lambda_X).$$

So, $\text{codim } \overline{\text{NL}(\Lambda_X)} = \dim \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}(d))) - \dim \overline{\text{NL}(\Lambda_X)} = h^0(\mathcal{O}_{\mathbb{P}^3}(d)) - \dim I_d(C) - \dim L_{\Lambda_X}$.

This finishes the proof of the proposition. \square

5 A Griffiths-Harris conjecture

5.1. We now come to the final section of the article, where we prove an asymptotic case of a Griffiths-Harris conjecture. Recall, a *Griffiths-Harris conjecture* states in [GH83] that:

For $3 \leq r \leq d$, the codimension of an irreducible component of $\text{NL}_{r,d}$ is at least equal to

$$(r-1)(d-3) - \binom{r-3}{2}.$$

Furthermore, there exists a component of $\text{NL}_{r,d}$ of this codimension parametrizing the space of surfaces containing $r-1$ lines on the same plane.

Notation 5.2. We will denote by $\mathcal{N}_d(r)$ the number,

$$(d-3)(r-1) - \binom{r-3}{2}.$$

We now recall a result in Noether-Lefschetz locus due to Otwinowska which will help us characterize the irreducible components of $\text{NL}_{r,d}$ with codimension less than or equal to $\mathcal{N}_d(r)$.

Theorem 5.3 ([Otw04, Theorem 1]). Let γ be an augmented lattice of rank 2 on a degree d surface. There exists $C \in \mathbb{R}_+^*$ depending only on r such that for $d \geq C(r-1)^8$ if $\text{codim } \text{NL}(\gamma) \leq$

$(r-1)d$ then $\gamma_{\text{prim}} = \sum_{i=1}^t a_i [C_i]_{\text{prim}}$, where $a_i \in \mathbb{Q}^*$, C_i are reduced curves and $\deg(C_i) \leq (r-1)$ for $i = 1, \dots, t$ for some positive integer t .

5.4. Throughout this section we denote by r an integer greater than or equal to 3 and for a fixed r , denote by d , an integer as mentioned in Theorem 5.3. We will *assume* that d is at least r^3 which will be used only in a computation in Lemma 5.9. The other results do not have any restriction on d in terms of r .

Proposition 5.5. Let L be an irreducible component of $\text{NL}_{r,d}$. Then L is locally homeomorphic to $\text{NL}(\Lambda)$ for some prime augmented lattice Λ of rank at least r on a surface $X \in L$, general.

Proof. Let $L \subset \text{NL}_{r,d}$ be an irreducible component. Let X be a general element in L . This implies that for the Picard lattice $\Lambda := \text{NS}(X)$, $\text{NL}(\Lambda)_{\text{red}}$ is an open subscheme of L_{red} . We can assume that Λ is a prime lattice. Since X is an element in $\text{NL}_{r,d}$, the rank of Λ is greater than or equal to r . \square

Proposition 5.6. If Λ is an augmented prime lattice of rank t on some degree d surface and $\text{codim } \overline{\text{NL}(\Lambda)} \leq (r-1)d$. Then there exists a prime lattice Λ' of rank greater than or equal to t generated by classes of curves of degree less than or equal to $r-1$, such that C_i deforms along $\text{NL}(\Lambda')$ and $\overline{\text{NL}(\Lambda)}_{\text{red}} = \overline{\text{NL}(\Lambda')}_{\text{red}}$.

Proof. Let $X \in \text{NL}(\Lambda)$. There exists a maximal lattice $\Lambda' \subset H^2(X, \mathbb{Z})$ such that Λ' remains of type $(1,1)$ in $\text{NL}(\Lambda)$ i.e., $\text{NL}(\Lambda)_{\text{red}} = \text{NL}(\Lambda')_{\text{red}}$. Now, there exists a surface $X' \in \text{NL}(\Lambda')$ such that the Néron-Severi group $\text{NS}(X')$ is the translate (under deformation from X to X') of Λ' in $H^2(X', \mathbb{Z})$ which we again denote by Λ' for convenience of notation. Then Theorem 5.3 implies that any $\gamma \in \Lambda'$ is of the form $\sum_i a_i [C_i] + bH_X$ with $\deg(C_i) \leq r-1$. So, Λ' can be generated by classes of curves of degree at most $r-1$ and the class of the very ample line bundle H_X . Now, the class of $[C_i]$ remains of type $(1,1)$ along $\text{NL}(\Lambda')$. From Lemma 3.4 it follows that C_i is semi-regular. Then, [Blo72, Theorem 7.1] implies that the class of $[C_i]$ remains effective along $\text{NL}(\Lambda')$. This proves the proposition. \square

5.7. We now recall a result due to Eisenbud and Harris which we use in the next lemma. Let P be a Hilbert polynomial of a curve in \mathbb{P}^3 of degree e and L be an irreducible component of H_P . The corollary after [EH92, Theorem 1] tells us that,

Theorem 5.8 ([EH92]). For $e > 1$, the dimension of L is less than or equal to $3 + e(e + 3)/2$.

Lemma 5.9. Let Λ be a prime augmented lattice of rank $t + 1$ on a degree d surface, generated by irreducible curves C_i for $i = 1, \dots, t$ for some positive integer t and $\deg(C_i) \leq r - 1$. Suppose $\text{codim NL}(\Lambda) \leq (r - 1)d$. Then, $\sum_{i=1}^t \deg(C_i) \leq (r - 1)$.

Proof. We prove this by induction on t . This is trivially true for $t = 1$. Suppose this is true for all $t \leq m$.

Assume this is not true for $t = m + 1$. In other words, there exists a prime lattice Λ minimally generated by $m + 1$ curves such that $\sum_i \deg(C_i) > r - 1$. This implies (after rearranging the indices if necessary) there exists an integer $0 < t' \leq m + 1$ such that $C_1, \dots, C_{t'}$ satisfies $\sum_{i=1}^{t'} \deg(C_i) > (r - 1)$ and $\sum_{i=1}^{t'-1} \deg(C_i) \leq (r - 1)$. Then, $\sum_{i=1}^{t'} \deg(C_i) \leq 2(r - 1)$. Denote by P the Hilbert polynomial of the curve $C_1 \cup \dots \cup C_{t'}$. We replace e by $2(r - 1)$ in Theorem 5.8 and conclude that the dimension of the Hilbert scheme H_P is less than or equal to $3 + (r - 1)(2r + 1)$. Using Proposition 4.6, the codimension of $\text{NL}([C_1 + \dots + C_{t'}])$ is greater than or equal to $\text{codim } I_d(C_1 \cup \dots \cup C_{t'}) - \dim H_P$. Since $r - 1 < \deg(C_1 + \dots + C_{t'}) \leq 2(r - 1)$ and $d \geq r^3$, we get the following inequality using the upper bound on the arithmetic genus of a curve of degree less than or equal to $2(r - 1)$:

$$\begin{aligned} \text{codim NL}([C_1 + \dots + C_{t'}]) &\geq \text{codim } I_d(C_1 + \dots + C_{t'}) - \dim H_P \\ &\geq (rd - (2r - 3)(2r - 4)/2 + 1) - (3 + (r - 1)(2r + 1)) \\ &= (r - 1)d + (d - (2r - 3)(2r - 4)/2 - 3 - (r - 1)(2r + 1) + 1) \\ &> (r - 1)d \end{aligned}$$

contradicting the assumption. □

Proposition 5.10. Let Λ be an augmented lattice of rank r contained in a degree d surface, generated by l_i for $i = 1, \dots, r - 1$, where l_i are lines for all i . Suppose $r \geq 3$. Then, $\text{codim NL}(\Lambda) \geq \mathcal{N}_d(r)$. Furthermore, if l_i are on the same plane we have an equality.

Proof. We prove this by induction. Using Proposition 4.6,

$$\text{codim NL}(\Lambda) = \text{codim } I_d\left(\bigcup_{i=1}^{r-1} l_i\right) - \dim L_\Lambda.$$

If $r = 3$, $\text{codim NL}(\Lambda) = 2d - 6 = \mathcal{N}_d(3)$. Assume the result holds true for all $r \leq m$ for some integer $m \geq 4$. We now prove for $r = m + 1$. Denote by Λ' the lattice generated by l_i for $i = 1, \dots, r - 2$ and by C the curve $\bigcup_{i=1}^{r-2} l_i$. Let $t := \dim L_\Lambda - \dim L_{\Lambda'}$. Note that, $t \leq 4$.

Note that $l_{r-1}.C \leq r - 2$. Denote by $\epsilon := r - 2 - l_{r-1}.C$. Comparing with the list of values of t , we see

1. If $\epsilon = 0$ then $l_{r-1}.C = r - 2$ and $t \leq 2$. In particular, for a fixed curve C there is a 1-1 correspondence between the set of choices of l_{r-1} intersecting C in $r - 2$ points and the set of planes P intersecting C at $r - 2$ distinct collinear points. If for a generic choice of P , $P \cap C$ are $r - 2$ distinct collinear points then all lines in C should lie on the same plane. In that case, l_{r-1} intersect C at $r - 2$ points if and only if l_{r-1} is on the same plane as C , hence $t = 2$ (dimension of the space of lines in \mathbb{P}^2). If this is not the case i.e., a generic choice of P does not intersect C in $r - 2$ distinct collinear points, then $t \leq 2$.
2. $\epsilon = r - 2$ and $t = 4$ if l_{r-1} does not intersect C .
3. $0 < \epsilon < r - 2$ and $t \leq 3$ otherwise.

Now,

$$\begin{aligned} \text{codim NL}(\Lambda) &= \text{codim } I_d\left(C \bigcup l_{r-1}\right) - \dim L_\Lambda \\ &= (\text{codim } I_d(C) + \text{codim } I_d(l_{r-1}) - l_{r-1}.C) - \dim L_\Lambda \\ &= (\text{codim } I_d(C) - \dim L_{\Lambda'}) + (\text{codim } I_d(l_{r-1}) - t) - (r - 2 - \epsilon). \end{aligned}$$

Writing $t = 4 - (4 - t)$ and using the induction step, we see that the right hand side of this

equation is greater than or equal to

$$\begin{aligned} & ((r-2)(d-3) - \binom{r-4}{2}) + (d-3) - (r-2-\epsilon) + (4-t) \\ &= ((r-2)(d-3) - \binom{r-3}{2}) + \epsilon - t + 2 \end{aligned}$$

Note that $\epsilon - t + 2 \geq 0$ from the three cases considered above. Substituting this inequality gives us $\text{codim NL}(\Lambda) \geq \mathcal{N}_d(r)$. \square

Theorem 5.11. Let L be an irreducible component of $\text{NL}_{r,d}$. Then, $\text{codim } L \geq (r-1)(d-3) - \binom{r-3}{2}$. Furthermore, there exists a component L of $\text{NL}_{r,d}$ of this codimension parametrizing surfaces containing $r-1$ coplanar lines.

Proof. It suffices to prove L is locally of the form $\text{NL}(\Lambda)$ for Λ a rank $r-1$ prime lattice generated by $[l_i]$ for $i = 1, \dots, r$, where l_i are lines on the same plane. Using Proposition 5.5, L is of the form $\text{NL}(\Lambda)$ for a rank $r-1$ prime lattice Λ . Using Proposition 5.6 we can assume that there exists a prime lattice Λ' of rank t greater than or equal to $r-1$ generated by classes of curves, say C'_i for $i = 1, \dots, t$ of degree less than or equal to $r-1$ such that $\text{NL}(\Lambda) = \text{NL}(\Lambda')$. Lemma 5.9 implies that $t = r-1$ and $\deg(C'_i) = 1$ for all $i = 1, \dots, r-1$. Finally, the theorem follows from Proposition 5.10. \square

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